Malaysian Journal of Mathematical Sciences 9(2): 175-186 (2015)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

The Inverse Problem for Pencils of Differential Operators on the Half-Line with Discontinuity

[1*]Abdolali Neamaty and [2]Yasser Khalili

[1]Faculty of Mathematical Science,
University of Mazandaran, Babolsar, Iran

[2]Department of Mathematics, Sari Branch,
Islamic Azad University, Sari, Iran

E-mail: namaty@umz.ac.ir and y.khalili@stu.umz.ac.ir

*Corresponding author

ABSTRACT

In this paper, we study second-order differential operators on the half-line having jump condition in an interior point. We obtain properties of the spectral characteristics, present a formulation of the inverse problem and prove the uniqueness theorem.

Keywords: Inverse problem, Jump condition, Differential pencil, Weyl function.

1. INTRODUCTION

We consider the boundary value problem L for the differential equation

$$y''(x) + \big(\rho^2 + i\rho p_1(x) + p_0(x)\big)y(x) = 0, \quad x \geq 0, \tag{1}$$

with the boundary condition

$$U(y) \coloneqq y'(0) + (\beta_1\rho + \beta_0)y(0) = 0, \tag{2}$$

and the jump condition

$$y(a-0) = \theta^{-1}y(a+0), \quad y'(a-0) = \theta y'(a+0), \tag{3}$$

in an interior point $a > 0$. Here $\theta \in (0,1)$, the coefficients β_0 and β_1 are complex numbers and $\beta_1 \neq \pm i$. The functions $P_k(x),\ k = 0,1$ are complex-valued, $P_1(x)$ is absolutely continuous and $(1+x)P_k^{(l)} \in L(0,\infty)$ for $0 \leq l \leq k \leq 1$.

Differential equations with a nonlinear dependence on the spectral parameter with discontinuities frequently appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences (see Keldysh, 1951; Kostyuchenko and Shkalikov, 1983; Markus, 1988; Tamarkin, 1917). For example, inverse spectral problems with discontinuous conditions arise for constructing parameters of heterogeneous electronic lines with favorable technical characteristics (see Meschanov and Feldstein, 1980). The boundary value problems without discontinuities have been studied in many works (see Levitan, 1987; Marchenko, 1986; McLaughlin, 1986; Neamaty and Khalili, 2013; Neamaty and Mosazadeh, 2011; Yurko, 2006). Indefinite differential equations with discontinuity construct considerable qualitative changes in the investigation of the inverse problem. In Freiling and Yurko, 2002 inverse problem has been considered for classical Sturm-Liouville operator with discontinuity. As far as we know the inverse problem for differential pencil with discontinuity has not been considered before. Some aspects of the inverse spectral problems for differential pencils have been studied in Gasymov and Gusejnov, 1981, Yamamoto, 1990 and other papers.

Inverse problems consist to obtain the coefficients of the boundary value problem from the certain spectral characteristics. Here we uniquely determine BVP(L) by its Weyl function. The rest of this paper is organized as follows: In the next section some results that are necessary in the sequel are provided. In Section 3, the uniqueness theorem is proved.

2. PRELIMINARY RESULTS AND WEYL FUNCTION

Denote $\Pi_{\pm} := \{\rho:\ \pm Im\rho > 0\}$ and $\Pi_0 := \{\rho:\ Im\rho = 0\}$. By the well-known method (see Yurko, 2006 and Freiling and Yurko, 2002), we get that for $x \geq a, \rho \in \Pi_{\pm}$ there exists a solution $y = e(x,\rho)$ of Eq.(1) (which is called the Jost-type solution) with the following properties:

(1) For each fixed $x \geq a$, the functions $e^{(v)}(x,\rho), v = 0,1$ are holomorphic for $\rho \in \Pi_+$ and $\rho \in \Pi_-$. Also the functions $e^{(v)}(x,\rho)$, $v = 0,1$ are continuous for $x \geq a, \rho \in \overline{\Pi_+}$ and $\rho \in \overline{\Pi_-}$. In other words, for real ρ, there exist the finite limits

$$e_{\pm}^{(v)}(x,\rho) = \lim_{z\to\rho,\ z\in\Pi_{\pm}} e^{(v)}(x,z).$$

Moreover, the functions $e^{(v)}(x,\rho), v = 0,1$ are continuously differentiable with respect to $\rho \in \overline{\Pi_+} \setminus \{0\}$ and $\rho \in \overline{\Pi_-} \setminus \{0\}$.

(2) For $x \to \infty, \rho \in \overline{\Pi_{\pm}} \setminus \{0\},\ v = 0,1,$

$$e^{(v)}(x,\rho) = (\pm i\rho)^v\, exp\left(\pm\big(i\rho x - Q(x)\big)\right)(1 + o(1)), \tag{4}$$

where

$$Q(x) = \frac{1}{2}\int_0^x p_1(t)dt. \tag{5}$$

(3) For $|\rho| \to \infty,\ \rho \in \overline{\Pi_{\pm}}$, uniformly in $x \geq a$,

$$e^{(v)}(x,\rho) = (\pm i\rho)^v\, exp\left(\pm\big(i\rho x - Q(x)\big)\right)[1], \tag{6}$$

where $[1] := 1 + O(\rho^{-1})$.

(4) For real $\rho \neq 0,\ x \in [0,\infty) \setminus \{a\}$ the functions $e_+(x,\rho)$ and $e_-(x,\rho)$ form a fundamental system of solutions of Eq.(1), and

$$\langle e_+(x,\rho), e_-(x,\rho)\rangle = -2i\rho,$$

where $\langle y,z\rangle \coloneqq yz' - y'z$, is the Wronskian of the functions y and z.

Denote

$$\Delta(\rho) \coloneqq U\big(e(x,\rho)\big). \tag{7}$$

The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem L. The function $\Delta(\rho)$ is holomorphic in Π_+ and Π_-, and for real ρ, there exist the finite limits

$$\Delta_{\pm}(\rho) = \lim_{z\to\rho,\ z\in\Pi_{\pm}} \Delta(z).$$

Moreover, the function $\Delta(\rho)$ is continuously differentiable for $\rho \in \overline{\Pi_{\pm}} \setminus \{0\}$.

Denote

$$\Lambda'_{\pm} = \{\rho \in \Pi_{\pm}:\ \Delta(\rho) = 0\},\quad \Lambda' = \Lambda'_+ \cup \Lambda'_-,$$
$$\Lambda''_{\pm} = \{\rho \in \mathbb{R}:\ \Delta_{\pm}(\rho) = 0\},\quad \Lambda'' = \Lambda''_+ \cup \Lambda''_-,$$

$$\Lambda_{\pm} = \Lambda'_{\pm} \cup \Lambda''_{\pm}, \ \Lambda = \Lambda_{+} \cup \Lambda_{-}.$$

Let $\varphi_j(x,\rho)$, $j = 1,2$, be the discontinuous solutions of Eq.(1) under the jump condition (3) and initial conditions $\varphi_j{}^{(n-1)}(0,\rho) = \delta_{jn}$, $n = 1,2$ (δ_{jn} is the Kronecker delta). For each fixed x, the functions $\varphi_j{}^{(n-1)}(x,\rho)$ are entire in ρ, and by virtue of Liouville's formula for the Wronskian, we have

$$\langle \varphi_1(x,\rho), \varphi_2(x,\rho) \rangle = 1. \tag{8}$$

We put (see Yurko, 2006)

$$\phi(x,\rho) = \frac{e(x,\rho)}{\Delta(\rho)}. \tag{9}$$

The function $\phi(x,\rho)$ is a solution of Eq.(1) that is called the Weyl solution of the boundary value problem L. Denote

$$M(\rho) = \phi(0,\rho). \tag{10}$$

We will call it the Weyl function for BVP(L). It follows from (9) and (10) that

$$M(\rho) = \frac{e(0,\rho)}{\Delta(\rho)}. \tag{11}$$

Using the initial conditions $\varphi(x,\rho)$ at the point $x = 0$, we get

$$\phi(x,\rho) = \varphi_2(x,\rho) + M(\rho)\varphi(x,\rho), \tag{12}$$

where

$$\varphi(x,\rho) = \varphi_1(x,\rho) - (\beta_1\rho + \beta_0)\varphi_2(x,\rho). \tag{13}$$

By virtue of (8), (12) and (13), we obtain

$$\langle \varphi(x,\rho), \phi(x,\rho) \rangle = 1. \tag{14}$$

Theorem 2.1. i_1. For $|\rho| \to \infty$, $\rho \in \overline{\Pi_{\pm}}$, the following asymptotical formula holds

$$\Delta(\rho) = \frac{\rho}{2\theta}\Big((\beta_1 + i)(1 \pm \theta^2)exp\big(-i\rho a + Q(a)\big)[1]$$

$$+(\beta_1 - i)(1 \mp \theta^2)exp\big(i\rho a - Q(a)\big)[1]\big)exp\left(\pm\big(i\rho a - Q(a)\big)\right). \quad (15)$$

i_2. For sufficiently large k, the function $\Delta(\rho)$ has zeros of the form

$$\rho_k = \frac{1}{a}\left(k\pi + \frac{Q(a)}{2i} + \kappa_1 + \kappa_2\right) + O(k^{-1}), \quad |k| \to \infty, \quad (16)$$

where

$$\kappa_1 = \frac{1}{2i} ln \frac{\beta_1 + i}{\beta_1 - i}, \qquad \kappa_2 = \frac{1}{2i} ln \frac{1 \pm \theta^2}{1 \mp \theta^2}.$$

i_3. The Weyl function $M(\rho)$ is holomorphic in $\Pi_\pm \backslash \Lambda'_\pm$ and continuously differentiable in $\overline{\Pi_\pm} \backslash \Lambda_\pm$. The set of singularities of $M(\rho)$ coincides with the set $\mathbb{R} \cup \Lambda = \{\rho: \rho \in \mathbb{R} \text{ or } \rho \in \Lambda\}$. For $|\rho| \to \infty$, $\rho \in \Pi_\pm^1$,

$$M(\rho) = \frac{1}{\rho(\beta_1 \mp i)}[1].$$

Proof. Denote $\Pi_\pm^1 := \{\rho: \pm Re\rho > 0\}$. Let $\{y_j(x,\rho)\}_{j=1,2}$ be the Birkhoff-type smooth fundamental system of solutions of Eq.(1) with the asymptotic forms

$$y_j^{(v)}(x,\rho) = ((-1)^{j-1} i\rho)^v exp\left((-1)^{j-1}\big(i\rho x - Q(x)\big)\right)[1], \; x \geq 0, \quad (17)$$

for $|\rho| \to \infty$, $\rho \in \overline{\Pi_\pm^1}$, $v = 0,1$, $j = 1,2$ (see Yurko, 2006; Freiling and Yurko, 2002). Then

$$e^{(v)}(x,\rho) = A_1(\rho) y_1^{(v)}(x,\rho) + A_2(\rho) y_2^{(v)}(x,\rho), \quad x \in [0,a]. \quad (18)$$

Using (6), (17), (18) and the jump condition (3), we calculate the coefficients $A_1(\rho)$ and $A_2(\rho)$ of the form

$$A_1(\rho) = \frac{1 \pm \theta^2}{2\theta} exp\big(-i\rho a + Q(a)\big)exp\left(\pm\big(i\rho a - Q(a)\big)\right)[1],$$
$$A_2(\rho) = \frac{1 \mp \theta^2}{2\theta} exp\big(i\rho a - Q(a)\big)exp\left(\pm\big(i\rho a - Q(a)\big)\right)[1].$$

Substituting the results into (18), we obtain

$$e^{(v)}(x,\rho)=\frac{(i\rho)^v}{2\theta}exp\left(\pm\left(i\rho a-Q(a)\right)\right)$$
$$\times\left((1\pm\theta^2)exp\left(i\rho(x-a)-\left(Q(x)-Q(a)\right)\right)[1]\right.$$
$$\left.+(-1)^v(1\mp\theta^2)exp\left(-i\rho(x-a)+\left(Q(x)-Q(a)\right)\right)[1]\right),\ \ x\in[0,a].$$

Together with (2) and (7), this yields (15). Now using (15) and the Rouche's theorem (see Conway, 1995), we obtain the zeros of the form (16). For proof of the part 3, it follows $M(\rho)$ from (11) and the functions $e(0,\rho)$ and $\Delta(\rho)$.

Remark 2.2. On can introduce the operator

$$\ell':D(\ell')\to\mathcal{L}_2(0,\infty),\quad y\to y''+\left(i\rho p_1(x)+p_0(x)\right)y,$$

with the domain of definition $D(\ell')=\{\mathrm{y}:\ \mathrm{y}\in\mathcal{L}_2(0,\infty)\cap\ \mathrm{AC}[0,a)\cap\mathrm{AC}_{\mathrm{loc}}(a,\infty),\quad \mathrm{y}'\in\mathrm{AC}[0,a)\cap\mathrm{AC}_{\mathrm{loc}}(a,\infty),\qquad \ell'\mathrm{y}\in\mathcal{L}_2(0,\infty),\mathrm{U(y)}=0,$ and $\mathrm{y}(x)$ satisfies (3)$\}$. It is easy to verify that the spectrum of ℓ' coincides with the spectrum of BVP(L). There is no difference between working with the operator ℓ' or BVP(L).

Definition 2.3. The set of singularities of the Weyl function $M(\rho)$ is called the spectrum of L (and is denoted by $\sigma(L)$). The values of the parameter ρ for which Eq.(1) has nontrivial solutions satisfying (2) and the condition $y(\infty)=0$ (i.e., $lim_{x\to\infty}y(x)=0$) are called the eigenvalues of L, and the corresponding solutions are called eigenfunctions.

Theorem 2.4. The BVP(L) has no eigenvalue for real $\rho\neq0$.

Proof. Suppose that $\rho_0\neq0$ is an eigenvalue, and $\mathrm{y}(\mathrm{x},\rho_0)$ be a corresponding eigenfunction. Since the functions $\{e_+(x,\rho_0),e_-(x,\rho_0)\}$ form a fundamental system of solutions for Eq.(1), we have $y(x,\rho_0)=C_1e_+(x,\rho_0)+C_2e_-(x,\rho_0)$. As $y(x,\rho_0)\cong0$ for $x\to\infty$, this is possible iff $C_1=C_2=0$. Therefore $\rho_0\neq0$ is not an eigenvalue.

Theorem 2.5. The functions $\varphi_j{}^{(v)}(x,\rho)$, $j=1,2,\ v=0,1$ have the following asymptotic forms for $|\rho|\to\infty$,

i_1. Uniformly in $x \in [0, a)$:

$$\begin{cases} \varphi_1{}^{(v)}(x,\rho) = \dfrac{(i\rho)^v}{2}\left(exp\left(i\rho x - Q(x)\right)[1] + (-1)^v exp\left(-i\rho x + Q(x)\right)[1]\right), \\ \varphi_2{}^{(v)}(x,\rho) = \dfrac{(i\rho)^v}{2i}\left(exp\left(i\rho x - Q(x)\right)O(\rho^{-1}) + (-1)^{v+1} exp\left(-i\rho x + Q(x)\right)O(\rho^{-1})\right). \end{cases}$$

i_2. Uniformly in $x \in (a, \infty)$:

$$\begin{cases} \varphi_1{}^{(v)}(x,\rho) = \dfrac{(i\rho)^v}{4\theta}\Big((\theta^2+1)\big(exp\big(i\rho x - Q(x)\big) \\ \qquad +(-1)^v exp\big(-i\rho x + Q(x)\big)\big)[1] \\ \qquad -(1-\theta^2)\Big(exp\Big(i\rho(x-2a) - \big(Q(x) - 2Q(a)\big)\Big) \\ \qquad +(-1)^v exp\Big(-i\rho(x-2a) + \big(Q(x) - 2Q(a)\big)\Big)\Big)[1]\Big), \\ \varphi_2{}^{(v)}(x,\rho) = \dfrac{(i\rho)^v}{4i\theta}\Big((\theta^2+1)\big(exp\big(i\rho x - Q(x)\big) \\ \qquad +(-1)^v exp\big(-i\rho x + Q(x)\big)\big)O(\rho^{-1}) \\ \qquad +(1-\theta^2)\Big(exp\Big(i\rho(x-2a) - \big(Q(x) - 2Q(a)\big)\Big) \\ \qquad +(-1)^v exp\Big(-i\rho(x-2a) + \big(Q(x) - 2Q(a)\big)\Big)\Big)O(\rho^{-1})\Big). \end{cases}$$

Proof. By Birkhoff-type smooth fundamental system of solutions of Eq.(1), we have

$$\varphi_j{}^{(v)}(x,\rho) = A_{1j}(\rho)y_1{}^{(v)}(x,\rho) + A_{2j}(\rho)y_2{}^{(v)}(x,\rho),\ j = 1,2, x \in [0,a]. \quad (19)$$

Using (17) and initial conditions $\varphi_j(x,\rho)$ in $x = 0$, we calculate

$$A_{1j}(\rho) = \begin{cases} \frac{1}{2}[1], & j = 1, \\ \frac{1}{2i}O(\rho^{-1}), & j = 2, \end{cases} \quad A_{2j}(\rho) = \begin{cases} \frac{1}{2}[1], & j = 1, \\ \frac{-1}{2i}O(\rho^{-1}), & j = 2. \end{cases} \quad (20)$$

Substituting (17) and (20) into (19), we obtain $\varphi_j{}^{(v)}(x,\rho)$ in $[0,a)$.

Analogously, taking the functions $\varphi_j{}^{(v)}(x,\rho)$ in $[0,a)$ and the jump condition (3), we arrive at $\varphi_j{}^{(v)}(x,\rho)$ in (a,∞).

Corollary 2.6. Using Theorem 2.5, we have for $x \geq 0$,

$$\begin{cases} \left|\varphi_1^{(v)}(x,\rho)\right| \le C|\rho|^v \, exp(|Im\rho|x), \\ \left|\varphi_2^{(v)}(x,\rho)\right| \le C|\rho|^{v-1} \, exp(|Im\rho|x), \\ \left|\varphi^{(v)}(x,\rho)\right| \le C|\rho|^v \, exp(|Im\rho|x). \end{cases} \tag{21}$$

In the next section, we will study the inverse problem for the boundary value problem L. The inverse problem is formulated as follows:

Inverse Problem 2.7. Given the Weyl function $M(\rho)$, construct the coefficients of the pencil (1)-(2).

3. THE UNIQUENESS THEOREM

In this section, we prove the uniqueness theorem for the solution of the inverse problem. For this purpose, together with $L = L(p_1(x), p_0(x), \beta_1, \beta_0)$, we will consider a boundary value problem $\tilde{L} = L(\tilde{p}_1(x), \tilde{p}_0(x), \tilde{\beta}_1, \tilde{\beta}_0)$ of the same form (1)-(2) but with different coefficients. If a certain symbol denotes an object related to L, then the corresponding symbol with tilde will denote the analogs object related to $\tilde{L}$.

Theorem 3.1. If $M(\rho) = \widetilde{M}(\rho)$ then $p_1(x) = \tilde{p}_1(x)$, $p_0(x) = \tilde{p}_0(x)$ for $x \ge 0$, and $\beta_1 = \tilde{\beta}_1$, $\beta_0 = \tilde{\beta}_0$. Thus the specification of the Weyl function uniquely determines the coefficients of the pencil (1)-(2).

Proof. At first, for brevity, we assume that a and θ are known a priori. By the assumption of Theorem 3.1 and the Weyl function, we infer $\beta_1 = \tilde{\beta}_1$.

Now we consider the matrix $P(x,\rho) = \left[P_{jk}(x,\rho)\right]_{j,k=1,2}$ defined by

$$P(x,\rho)\begin{bmatrix} \tilde{\varphi}(x,\rho) & \tilde{\phi}(x,\rho) \\ \tilde{\varphi}'(x,\rho) & \tilde{\phi}'(x,\rho) \end{bmatrix} = \begin{bmatrix} \varphi(x,\rho) & \phi(x,\rho) \\ \varphi'(x,\rho) & \phi'(x,\rho) \end{bmatrix}. \tag{22}$$

By virtue of (14), this yields

$$\begin{cases} P_{j1}(x,\rho) = \varphi^{(j-1)}(x,\rho)\tilde{\phi}'(x,\rho) - \phi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho), \\ P_{j2}(x,\rho) = \phi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{\phi}(x,\rho). \end{cases} \tag{23}$$

Also we have

$$\begin{cases}\varphi(x,\rho)=P_{11}(x,\rho)\tilde{\varphi}(x,\rho)+P_{12}(x,\rho)\tilde{\varphi}'(x,\rho),\\ \phi(x,\rho)=P_{11}(x,\rho)\tilde{\phi}(x,\rho)+P_{12}(x,\rho)\tilde{\phi}'(x,\rho).\end{cases} \tag{24}$$

Using (12) and (23), we calculate

$$\begin{cases}P_{j1}(x,\rho)=\varphi^{(j-1)}(x,\rho)\tilde{\varphi}_2'(x,\rho)-\varphi_2{}^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho)\\ \qquad\qquad +\widehat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho),\\ P_{j2}(x,\rho)=\varphi_2{}^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho)-\varphi^{(j-1)}(x,\rho)\tilde{\varphi}_2(x,\rho)\\ \qquad\qquad -\widehat{M}(\rho)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho),\end{cases}$$

where $\widehat{M}(\rho)=\widetilde{M}(\rho)-M(\rho)$. Since $\widetilde{M}(\rho)=M(\rho)$ deduce $\widehat{M}(\rho)=0$ and consequently, the functions $P_{jk}(x,\rho),\ k=1,2$ are entire in ρ for each fixed $x\geq 0$.

Let fix $\varepsilon>0$. Denote $G_\varepsilon=\{\rho\in\mathbb{C}:|\rho-\rho_k|\geq\varepsilon,\ \rho_k\in\Lambda\}$. It follows from (6), (11), (12), (15) and (21) and the function $e(x,\rho)$ in proof of the Theorem 2.1 that

$$\left|e^{(v)}(x,\rho)\right|\leq C|\rho|^v\, exp(-|Im\rho|x),\ \ x\geq 0,\ \ \rho\in\overline{\Pi_\pm}, \tag{25}$$

$$|\Delta(\rho)|\geq C|\rho|,\quad \rho\in G_\varepsilon, \tag{26}$$

$$\left|\phi^{(v)}(x,\rho)\right|\leq C|\rho|^{v-1}\, exp(-|Im\rho|x),\ x\geq 0,\ \rho\in G_\varepsilon. \tag{27}$$

It follows from (21), (23) and (27) that for $x\geq 0,\rho\in G_\varepsilon$,

$$|P_{11}(x,\rho)|\leq C,\qquad\qquad |P_{12}(x,\rho)|\leq C|\rho|^{-1}.$$

Therefore $P_{11}(x,\rho)=P_1(x)$ and $P_{12}(x,\rho)=0$ for each $x\geq 0$. Together with (24), we have for all x,ρ that

$$P_1(x)\tilde{\varphi}(x,\rho)=\varphi(x,\rho),\ \ P_1(x)\tilde{\phi}(x,\rho)=\phi(x,\rho). \tag{28}$$

First let $x\in[0,a]$. Taking the functions $\varphi_j(x,\rho), j=1,2$ and $e(x,\rho)$ in $[0,a]$, (9), (13), (15) and equality $\beta_1=\tilde{\beta}_1$, we get as $|\rho|\to\infty$, arg $\rho\in\left(0,\frac{\pi}{2}\right)$,

$$\begin{cases} \frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = exp\left(Q(x) - \tilde{Q}(x)\right)[1], \\ \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = exp\left(-\left(Q(x) - \tilde{Q}(x)\right)\right)[1]. \end{cases} \tag{29}$$

One has from (28) and (29) that

$$P_1(x) = exp\left(Q(x) - \tilde{Q}(x)\right)[1], \quad P_1(x) = exp\left(-\left(Q(x) - \tilde{Q}(x)\right)\right)[1], \tag{30}$$

and consequently, $Q(x) = \tilde{Q}(x)$ and $P_1(x) = 1$ for $x \in [0, a)$.

Let $x > a$. Taking the functions $\varphi_j(x,\rho), j = 1,2$ and $e(x,\rho)$ as $x > a$, (9), (13), (15) and equalities $\beta_1 = \tilde{\beta}_1$ and $Q(a) = \tilde{Q}(a)$ into accounts, we have as $|\rho| \to \infty$, $\arg \rho \in \left(0, \frac{\pi}{2}\right)$,

$$\begin{cases} \frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = exp\left(Q_a(x) - \widetilde{Q_a}(x)\right)[1], \\ \frac{\phi(x,\rho)}{\tilde{\phi}(x,\rho)} = exp\left(-\left(Q_a(x) - \widetilde{Q_a}(x)\right)\right)[1], \end{cases} \tag{31}$$

where $Q_a(x) = \frac{1}{2}\int_a^x p_1(t)dt$. It follows from (28) and (31) that

$$P_1(x) = exp\left(Q_a(x) - \tilde{Q}_a(x)\right)[1], \; P_1(x) = exp\left(-\left(Q_a(x) - \tilde{Q}_a(x)\right)\right)[1]. \tag{32}$$

Therefore $Q_a(x) = \tilde{Q}_a(x)$ and $P_1(x) = 1$ for $x > a$.

Thus $p_1(x) = \tilde{p}_1(x)$ and $P_1(x) = 1$ for all $x \geq 0$. According to (28), we have

$$\tilde{\varphi}(x,\rho) = \varphi(x,\rho), \quad \tilde{\phi}(x,\rho) = \phi(x,\rho) \tag{33}$$

Hence $p_0(x) = \tilde{p}_0(x)$ on $(0,\infty)$ and $\beta_0 = \tilde{\beta}_0$. The proof of Theorem 3.1 is completed.

ACKNOWLEDGEMENT

The authors express their thanks to the anonymous reviewers for their valuable comments.

REFERENCES

Conway, J.B. (1995). *Functions of One Complex Variable*. Vol. I. New York: Springer.

Freiling, G. and Yurko, V. (2002). Inverse spectral problems for singular non-self adjoint differential operators with discontinuities in an interior point.*Inverse problems*. **18**: 757–773.

Gasymov, M.G. and Gusejnov, G.Sh. (1981). Determination of a diffusion operator from spectral data.*Dokl. Akad. Nauk Azerb. SSR*.**37**: 19-23.

Keldysh, M.V. (1951). On eigenvalues and eigenfunctions of some classes of non-self adjoint equations.*Dokl. Akad. Nauk Azerb. SSR*.**77**: 11-14.

Kostyuchenko, A.G. and Shkalikov, A.A. (1983). Self adjoint quadratic operator pencils and elliptic problems.*Funct. Anal. Appl*. **17**: 109-128.

Levitan, B.M. (1987). *Inverse Sturm-Liouville Problems*. Utrecht: VNU Sci. Press.

Marchenko, V.A. (1977). *Sturm-Liouville Operators and Their Applications*.Kiev: Naukova Dumka.

Markus, A.S. (1988). *Introduction to the Spectral Theory of Polynomial Operator Pencils*.Providence, Rhode Island: AMS.

McLaughlin, J.R. (1986). Analytical methods for recovering coefficients in differential equations from spectral data.*SIAM Rev*. **28**: 53-72.

Meschanov, V.P. and Feldstein, A.L. (1980). *Automatic Design of Directional Couplers*. Moscow: Sviaz.

Neamaty, A. and Khalili, Y. (2013). The differential pencils with turning point on the half line.*Arab J. Math . Sci*. **19**(1): 95-104.

Neamaty, A. and Mosazadeh, S. (2011). On the canonical solution of the Sturm-Liouville problem with singularity and turning point of even order.*Canad. Math. Bull*. **54**(3): 506-518.

Tamarkin, J.D. (1917). *On Some Problems of the Theory of Ordinary Linear Differential Equations*. Petrograd.

Yamamoto, M. (1990). Inverse eigenvalue problem for a vibration of a string with viscous drag. *J. Math. Anal. Appl*. **152**: 20-34.

Yurko, V. (2006). Inverse spectral problems for differential pencils on the half-line with turning points. *J. Math. Anal*. **320**: 439-463.